Analytical procedure to determine the self-referred lacunarity function for simple shapes. Supplementary material.

Erbe P. Rodrigues, Marconi. S. Barbosa and Luciano da F. Costa
Instituto de Física de São Carlos
Universidade de São Paulo Caixa Postal 369 CEP 13560.970 São Carlos, SP, Brazil
e-mail: marconi@if.sc.usp.br

Abstract

The analytical calculation of the self-referred lacunarity is used as a validation standard of the computational algorithm. In this supplementary material to our article (see cond-mat/0407079) we present a detailed calculation for two simple shapes, namely a square box and a cross.

1 Introduction: self-referred lacunarity

The lacunarity of an object is defined as

$$\Lambda(r) = \frac{Z^{(2)}}{[Z(1)]^2} = \frac{\sigma^2(r)}{[\mu(r)]^2} + 1,$$

where $Z^{(1)}$ and $Z^{(2)}$ are the first and second moments of a mass distribution and $\sigma^2(r)$ and $\mu(r)$ are its variance and mean. We can calculate $\mu(r)$ and $\sigma^2(r)$ by defining a function $A(x, y, r)$ that gives the area of the object inside a window of radius r, whose center is in the point (x, y), in the following way

$$\mu(r) = \frac{\int_x \int_y A(x, y, r)dxdy}{\int_x \int_y dxdy}, \tag{1}$$

$$\sigma^2(r) = \frac{\int_x \int_y [\mu(r) - A(x, y, r)]^2 dxdy}{\int_x \int_y dxdy}. \tag{2}$$

For the self-referred approach of lacunarity the integral is calculated only for values of x and y that fall over the object, see Figure 1.

2 Analytical calculation for simple shapes

2.1 Object: a square box

We first show how to determine the self-referred lacunarity for a solid square of side L shown in Figure 1(a). In order to calculate the analytical lacunarity of a square it is necessary to determine the area funcition $A(x, y, r)$. This function is determined by dividing the square in regions labeled by I, II and III as in Figure 1(a) and 1(b). There is one region of type I and four of type II and III. It is also necessary to divide this calculation into two intervals: $r = [0, L/2]$ and $r = [L/2, L]$.

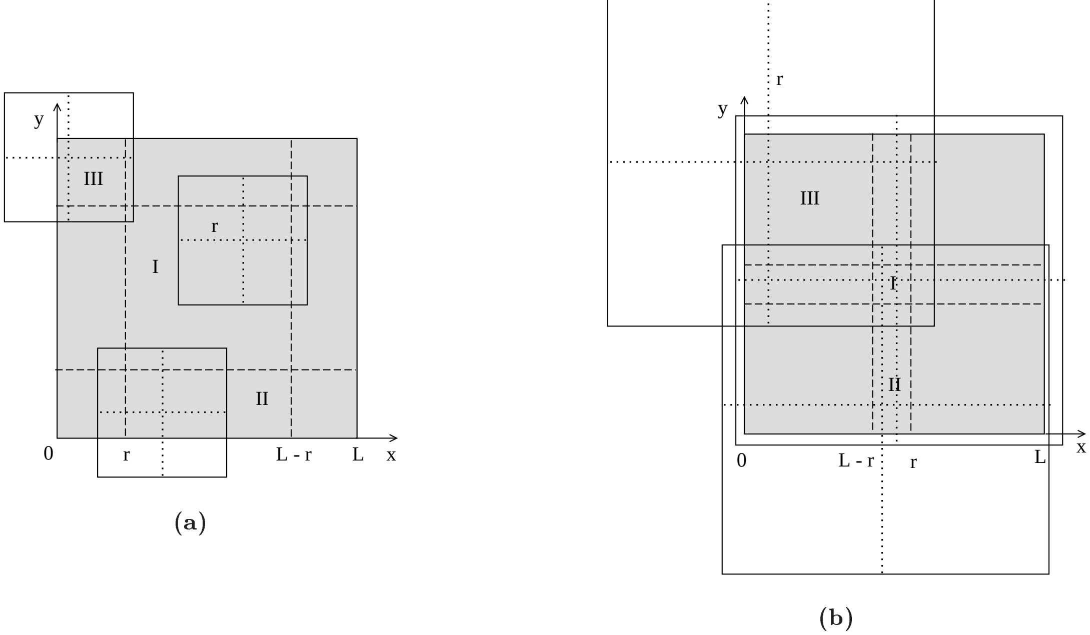

Figure 1: Square regions that correspond to the self-referred lacunarity calculation for the intervals (a) $r \in [0, L/2]$ and (b) $r \in [L/2, L]$. The sliding windows in the region I are completely filled in (a) and capture all the square in (b). In the regions II and III they are partially filled in (a) as well as in (b).

For the first interval, $r \in [0, L/2]$, the functions $A_i^1(x, y, r)$ and respective integration ranges for x and y are

$$\begin{aligned}
A_I^1(x,y,r) &= 4r^2, \quad x \in [r,(L-r)], \quad \in [r,(L-r)] \\
A_{II}^1(x,y,r) &= 2r^2 + 2ry, \quad x \in [r,(L-r)], \quad \in [0,r] \\
A_{III}^1(x,y,r) &= r^2 + [r + (L-y)]x + r(L-y), \\
& \quad x \in [0,r], \quad y \in [(L-r),r]
\end{aligned}$$

The mean value is determined by the integration of the A_i^1 over their respective regions divided by the total area of the square, according the Equation 1

$$\begin{aligned}
\mu_1(r) &= \frac{\int_x \int_y \sum_i A_i^1(x,y,r)dxdy}{\int_x \int_y dxdy} \\
&= \frac{1}{L^2} \int \int A_I^1(x,y,r) + 4A_{II}^1(x,y,r) + 4A_{III}^1(x,y,r)dxdy \\
&= \frac{r^2(r-2L)^2}{L^2}
\end{aligned}$$

The four regions II and III are equivalent and it is necessary only to integrate over one of region II and III and multiply the result by 4. The variance is calculated in analogous way according to Equation 2

$$\begin{aligned}
\sigma_1^2(r) &= \frac{1}{L^2} \int \int (A_I^1(x,y,r) - \mu_1(r))^2 \\
&+ 4(A_{II}^1(x,y,r) - \mu_1(r))^2 \\
&+ 4(A_{III}^1(x,y,r) - \mu_1(r))^2 dxdy \\
&= -\frac{r^5(r-6L)(3r-4L)(3r-2L)}{9L^4}
\end{aligned} \tag{3}$$

interval

$$\Lambda_1(r) = \frac{\sigma_1^2(r)}{[\mu_1(r)]^2} + 1 = \frac{4L^2(5r-6L)^2}{9(r-2L)^4}$$

The calculation for the second interval, $r \in [L/2, L]$, proceeds in the same way, with the A_i^2 functions and their respectives integration intervals given by

$$\begin{aligned}
A_I^2(x,y,r) &= L^2, \quad x \in [(L-r), r], \quad y \in [(L-r), r] \\
A_{II}^2(x,y,r) &= L(r+y), \quad x \in [(L-r), r], \quad y \in [0, (L-r)] \\
A_{III}^2(x,y,r) &= r^2 + [r + (L-y)]x + r(L-y), \\
& \quad x \in [0, (L-r)], \quad y \in [r, L]
\end{aligned}$$

and the mean, variance and Lacunarity as in

$$\begin{aligned}
\mu_2(r) &= \frac{r^2(r-2L)^2}{L^2} \\
\sigma_2^2(r) &= -\frac{(3L-r)(r-L)^3(3r^4 - 14r^3L + 12r^2L^2 + 6rL^3 - L^4)}{9L^4} \\
\Lambda_2(r) &= \frac{L^2(2r^3 - 6rL^2 + L^3)^2}{9r^4(r-2L)^4}
\end{aligned}$$

The final expression for the whole interval self-referred lacunarity is

$$\Lambda(r) = \begin{cases} \frac{4L^2(5r-6L)^2}{9(r-2L)^4}, & r \in [0, L/2]; \\ \\ \frac{L^2(2r^3-6rL^2+L^3)^2}{9r^4(r-2L)^4}, & r \in [L/2, L]. \end{cases}$$

Figure 2 shows a plot of the expression above for a square of length $L = 100$, in absolute units of length.

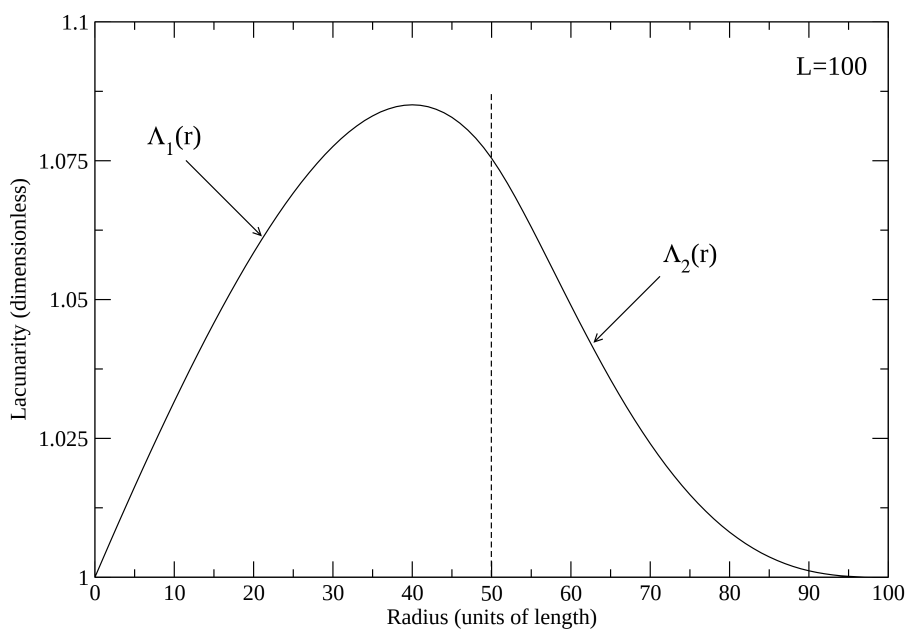

Figure 2: The analytically calculated self-referred lacunarity curve for a square with side L of 100 absolute units of length.

The analytical procedure to calculate the lacunarity of a cross, which occupies a region k, is similar to the procedure presented above for a square. Again we define a function $A(x, y, r)$. For this case, there are seven intervals and each one was divided into regions which define an associated function $A_i(x, y, r)$. Figure 3 shows the first interval considered in the self-referred lacunarity calculation of a cross with dimensions $L1$ and $L2$ and the respective regions. Figure 3 presents only non-repeated regions i, however the calculation regards all the possible regions j. In the following we show the calculation of the first interval in more detail. For the remaining intervals we provide only the functions $A_i(x, y, r)$, $\mu(r)$, $\sigma^2(r)$ and $\Lambda(r)$. Figures 4 to 9 presents the divided regions for the remaining intervals labeled from 2 to 7. The expression for lacunarity, $\Lambda(r) = (\sigma^2(r)/\mu^2(r)) + 1$ where μ and σ^2 is given by

$$\mu(r) = \frac{\int_k \sum A_j(x, y, r) dk}{\int_k dk},$$

and

$$\sigma^2(r) = \frac{\int_k \left[\sum A_j(x, y, r) - \mu(r)\right]^2 dk}{\int_k dk},$$

where $\int_k dk$ is the total k area ($A_k = 4L1L2 + L2^2$) of the cross.

2.2.1 Interval $0 \leq r \leq \frac{L2}{2}$

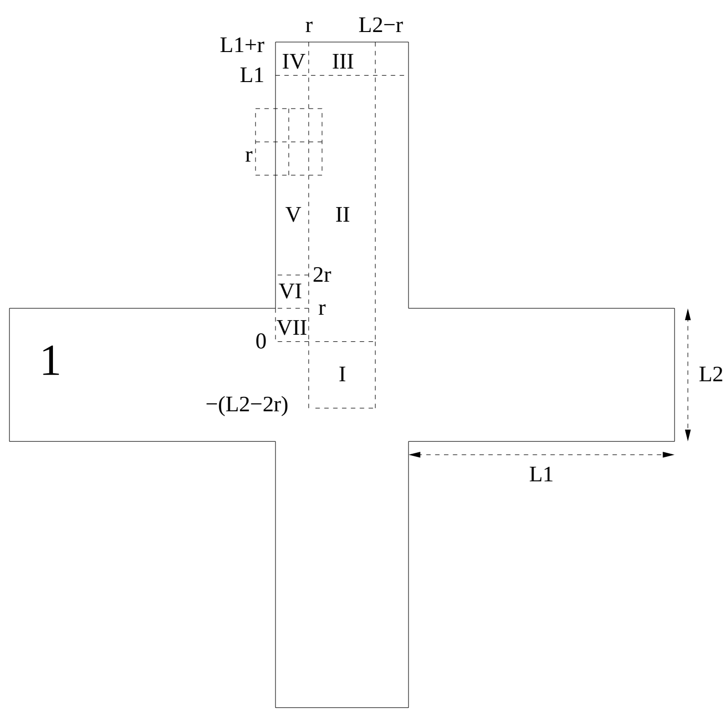

Figure 3: The cross divided to regions for the first interval.

$$\begin{aligned}
A_I(x, y, r) &= 4r^2 \quad x \in [r, L2 - r], \quad y \in [-(L2 - 2r), 0] \\
A_{II}(x, y, r) &= 4r^2 \quad x \in [r, L2 - r], \quad y \in [0, L1] \\
A_{III}(x, y, r) &= (2r + L1 - y)2r \quad x \in [r, L2 - r], \quad y \in [L1, L1 + r]
\end{aligned}$$

$$
\begin{aligned}
A_V(x,y,r) &= (r+x)(2r) \quad x\in[0,r], \quad y\in[2r,L1]\\
A_{VI}(x,y,r) &= 4r^2-y(r-x) \quad x\in[0,r], \quad y\in[r,2r]\\
A_{VII}(x,y,r) &= 4r^2-y(r-x) \quad x\in[0,r], \quad y\in[0,r]
\end{aligned}
$$

The Calculation of the mean μ for the first region gives

$$
\begin{aligned}
\mu_I(r) &= \frac{\int_k (A_I+4A_{II}+4A_{III}+8A_{IV}+8A_V+8A_{VI}+4A_{VII})dk}{A_k}\\
&= \frac{r^2(4L2^2-4rL2+3r^2+16L1L2-8rL1)}{L2(4L1+L2)}.
\end{aligned}
$$

We make $L1=2L$ e $L2=L$ to simplify the expressions. Thus we have for the previous expression

$$
\mu_I(r)=\frac{r^2(36l^2-20Lr+3r^2)}{9L^2}.
$$

In an analogous way $\sigma^2(r)$ is given by

$$
\begin{aligned}
\sigma_I^2(r) &= \frac{\int_k \left([A_I-\mu_I(L1,L2,r)]^2+\ldots+[4A_{VII}-\mu_I(L1,L2,r)]^2\right)dk}{A_k}\\
&= -\frac{1}{9L2^2(4L1+L2)^2}(-384L2L1^2+576rL1^2-432r^2L1-288L1L2^2\\
&\quad +496rL1L2-216L2r^2-48L2^3+81r^3+124rL2^2)r^5.
\end{aligned}
$$

Simplifying the above expression we have

$$
\sigma_I^2(r)=-\frac{r^5\left(9r^3-120Lr^2+380rL^2-240L^3\right)}{81L^4}.
$$

The lacunarity expression in this interval is finally given by

$$
\Lambda_I(r)=\frac{4L^2(-300Lr+59r^2+324L^2)}{(36L^2-20Lr+3r^2)^2}.
$$

2.2.2 Interval $\frac{L2}{2}\le r\le L2$

$$
\begin{aligned}
A_I(x,y,r) &= 2(2rL2)-L2^2 \quad x\in[L2-r,r], y\in[-(L2+2r),0]\\
A_{II}(x,y,r) &= (2r+y-L2)L2+(L2-y)2r \quad x\in[L2-r,r], \quad y\in[0,L2]\\
A_{III}(x,y,r) &= L2\times 2r \quad x\in[L2-r,r], \quad y\in[L2,L1+L2-2r]\\
A_{IV}(x,y,r) &= L2(L1+L2-y) \quad x\in[L2-r,r],\\
&\quad\ y\in[L1+L2-2r,L1+L2-r]\\
A_V(x,y,r) &= (L1+L2-y)(r+x) \quad x\in[0,L2-r],\\
&\quad\ y\in[L1+L2-2r,L1+L2-r]\\
A_{VI}(x,y,r) &= 2r(r+x) \quad x\in[0,L2-r], \quad y\in[L2,L1+L2-2r]\\
A_{VII}(x,y,r) &= 2r(r+x)+(r-x)(L2-y) \quad x\in[0,L2-r], y\in[L2-r,L2]\\
A_{VIII}(x,y,r) &= 4r^2-(2r+y-L2)(r-x) \quad x\in[0,L2-r], \quad y\in[0,L2-r]
\end{aligned}
$$

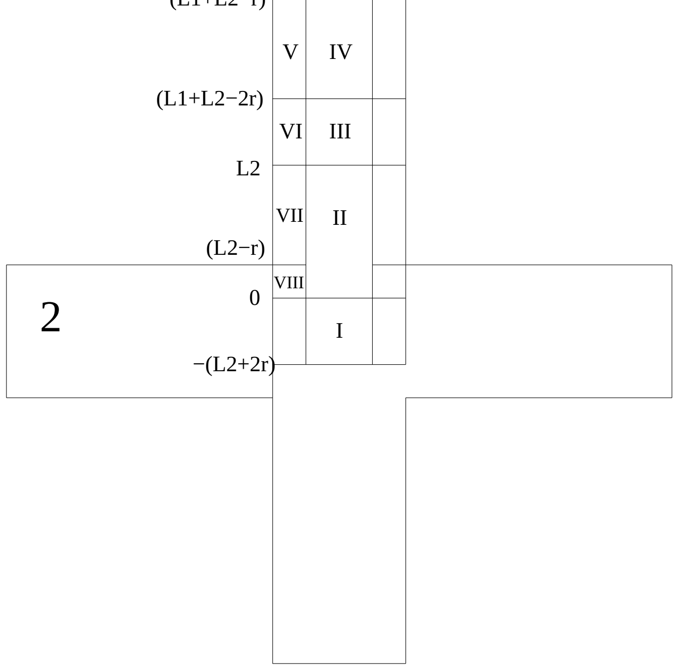

Figure 4: The cross regions to interval $\frac{L2}{2} \leq r \leq L2$.

$$\mu_{II}(r) = \frac{r^2(-20rL + 36L^2 + 3r^2)}{9L^2}$$

$$\begin{aligned}\sigma^2_{II}(r) &= \frac{1}{81L^4}(120r^7L - 12L^7r - 9r^8 + 1248r^5L^3 + 452L^5r^3 \\ &+L^8 - 36L^6r^2 - 604r^6L^2 - 1128L^4r^4)\end{aligned}$$

$$\Lambda_{II}(r) = \frac{L^2\left(-12\,L^5r - 192\,r^5L + 452\,L^3r^3 + L^6 - 36\,L^4r^2 + 12\,r^6 + 168\,r^4L^2\right)}{r^4\left(-20\,rL + 36\,L^2 + 3\,r^2\right)^2}$$

2.2.3 Interval $L2 \leq r \leq \frac{L1+L2}{2}$

$$\begin{aligned}A_I(x,y,r) &= 2L2(2r) - L2^2 \quad x \in [0, L2], \quad y \in [-L2, 0] \\ A_{II}(x,y,r) &= 4rL2 - L2^2 \quad x \in [0, L2], \quad y \in [0, r - L2] \\ A_{III}(x,y,r) &= L2(r+y) + (r-y)2r \quad x \in [0, L2], \quad y \in [r - L2, L1 - r] \\ A_{IV}(x,y,r) &= (L1L2) + (r-y)2r \quad x \in [0, L2], \quad y \in [L1 - r, r)] \\ A_V(x,y,r) &= (L1 - (y-r))L2 \quad x \in [0, L2], \quad y \in [r, L1]\end{aligned}$$

$$\mu_{III}(r) = \frac{4L}{3}rL + \frac{1}{9}L^2 + \frac{2}{3}r^2$$

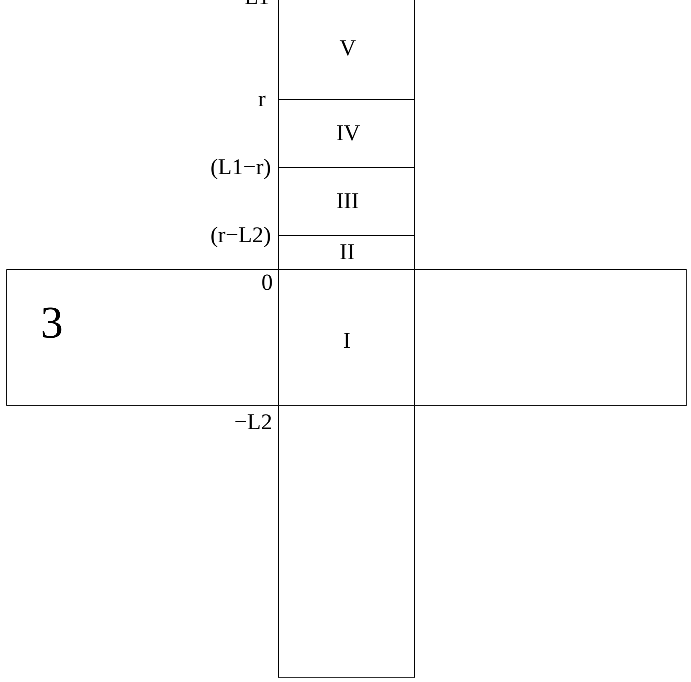

Figure 5: The cross regions for the interval $L2 \leq r \leq \frac{(L1+L2)}{2}$.

$$\sigma^2_{III}(r) = -\frac{76}{27}\,r^4 - \frac{124}{9}\,r^2L^2 + \frac{100}{9}\,Lr^3 + \frac{196}{27}\,L^3r - \frac{112}{81}\,L^4$$

$$\Lambda_{III}(r) = -3\,\frac{64\,r^4 + 320\,r^2L^2 - 348\,Lr^3 - 204\,L^3r + 37\,L^4}{\left(12\,rL + L^2 + 6\,r^2\right)^2}$$

2.2.4 Interval $\frac{L1+L2}{2} \leq r \leq L1$

$$\begin{aligned}
A_I(x,y,r) &= 2L2(2r) - L2^2 \quad x \in [0, L2], \quad y \in [-L2, 0] \\
A_{II}(x,y,r) &= 2rL2 + (r-y-L2)L2 + (y+r)L2 \quad x \in [0, L2], \quad y \in [0, L1-r] \\
A_{III}(x,y,r) &= L1L2 + 2rL2 + (r-y-L2)L2 \quad x \in [0, L2], \quad y \in [L1-r, r-L2] \\
A_{IV}(x,y,r) &= L1L2 + (r-y)2r \quad x \in [0, L2], \quad y \in [r-L2, r)] \\
A_V(x,y,r) &= (L1-(y-r))L2 \quad x \in [0, L2], \quad y \in [r, L1]
\end{aligned}$$

$$\mu_{IV}(r) = \frac{4L}{3}rL + \frac{1}{9}L^2 + \frac{2}{3}r^2$$

$$\sigma^2_{IV}(r) = -\frac{4}{9}\,r^4 + \frac{68}{9}\,r^2L^2 - \frac{20}{27}\,Lr^3 - \frac{236}{27}\,L^3r + \frac{212}{81}\,L^4$$

$$\Lambda_{IV}(r) = 3\,\frac{L\left(256\,r^2L + 28\,r^3 - 228\,rL^2 + 71\,L^3\right)}{\left(12\,rL + L^2 + 6\,r^2\right)^2}$$

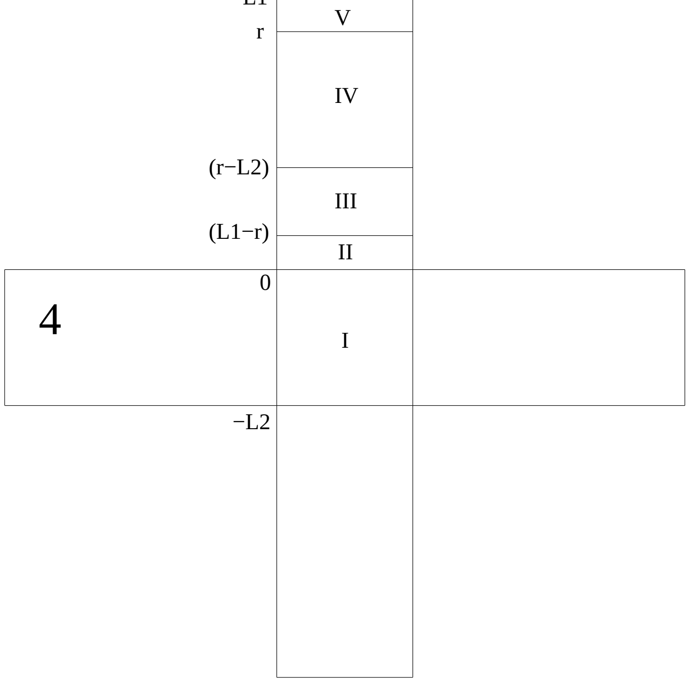

Figure 6: The cross regions for the interval $\frac{(L1+L2)}{2} \leq r \leq L1$.

2.2.5 Interval $L1 \leq r \leq L1 + \frac{L2}{2}$

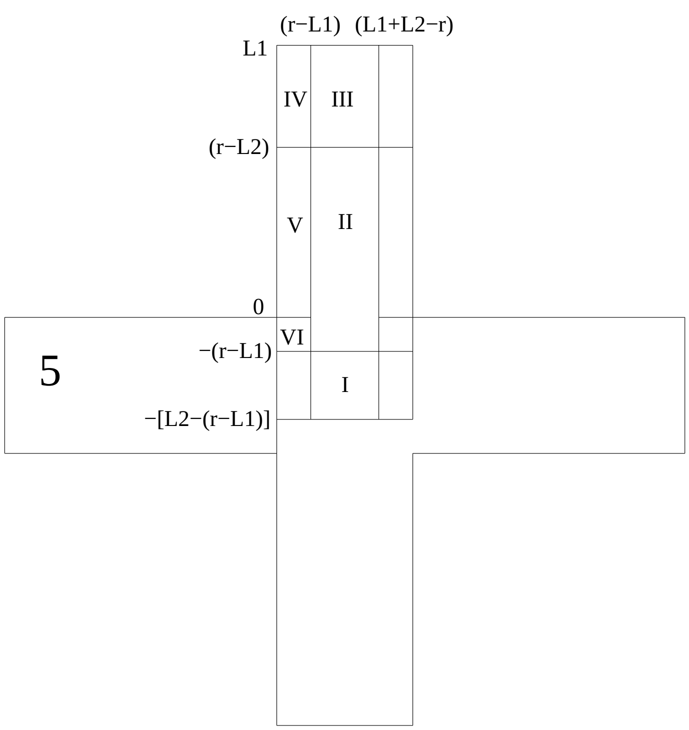

Figure 7: The cross regions for the interval $L1 \leq r \leq L1 + \frac{L2}{2}$.

$$
\begin{aligned}
A_I(x,y,r) &= 2(2rL2) - L2^2 \quad x \in [r-L1, L1+L2-r], \\
& \quad y \in [-L1-L2+r, -r+L1] \\
A_{II}(x,y,r) &= L1L2 + 2rL2 + (r-y-L2)L2 \quad x \in [r-L1, L2+L1-r], \\
& \quad y \in [-r+L1, r-L2] \\
A_{III}(x,y,r) &= L1L2) + (r-y)2r \quad x \in [r-L1, L2+L1-r], \quad y \in [r-L2, L1] \\
A_{IV}(x,y,r) &= L1L2 + (r-y)(r+x+L1) \quad x \in [0, r-L1], \quad y \in [r-L2, L1] \\
A_V(x,y,r) &= L1L2 + (r+x+L1)L2 + (r-y-L2)L2 \quad x \in [0, r-L1], \\
& \quad y \in [0, r-L2] \\
A_{VI}(x,y,r) &= L1L2 + (r+x+L1)L2 + (r-y-L2)L2 \quad x \in [0, r-L1], \\
& \quad y \in [-(r-L1), 0]
\end{aligned}
$$

$$
\mu_V(r) = \frac{-100\,L^3 r + 90\,r^2 L^2 - 24\,L r^3 + 2\,r^4 + 49\,L^4}{9L^2}
$$

$$
\begin{aligned}
\sigma_V^2(r) &= -\frac{2}{81L^4}(398\,L^8 + 6704\,L^6 r^2 + 5717\,L^4 r^4 + 448\,r^6 L^2 \\
& \quad -2144\,r^5 L^3 - 8516\,L^5 r^3 - 2542\,L^7 r - 48\,r^7 L + 2\,r^8)
\end{aligned}
$$

$$
\Lambda_V(r) = \frac{L^2\left(1605\,L^6 + 5412\,L^4 r^2 + 1662\,r^4 L^2 + 40\,r^6 - 432\,r^5 L - 3320\,L^3 r^3 - 4716\,L^5 r\right)}{\left(-100\,L^3 r + 90\,r^2 L^2 - 24\,L r^3 + 2\,r^4 + 49\,L^4\right)^2}
$$

2.2.6 Interval $L1 + \frac{L2}{2} \le r \le L1 + L2$

$$
\begin{aligned}
A_I(x,y,r) &= 4L1L2 + L2^2 \quad x \in [L1+L2-r, r-L1], \\
& \quad y \in [-(r-L1), -(L1+L2-r)] \\
A_{II}(x,y,r) &= L2^2 + 3L1L2 + (r-y-L2)L2 \quad x \in [-(L2+L1-r), r-L1], \\
& \quad y \in [-(L1+L2-r), r-L2] \\
A_{III}(x,y,r) &= L1L2 + (r-y)(2L1+L2) \quad x \in [L2+L1-r, r-L1], \\
& \quad y \in [r-L2, L1] \\
A_{IV}(x,y,r) &= L1L2 + (L1+r+x)(r-y) \quad x \in [0, L2+L1-r], \quad y \in [r-L2, L1] \\
A_V(x,y,r) &= L1L2 + (r-y-L2)L2 + (L1+r+x)L2 \quad x \in [0, L2+L1-r], \\
& \quad y \in [0, r-L2] \\
A_{VI}(x,y,r) &= L1L2 + (r-y-L2)L2 + (L1+r+x)L2 \quad x \in [0, L2+L1-r], \\
& \quad y \in [-(L1+L2-r), 0]
\end{aligned}
$$

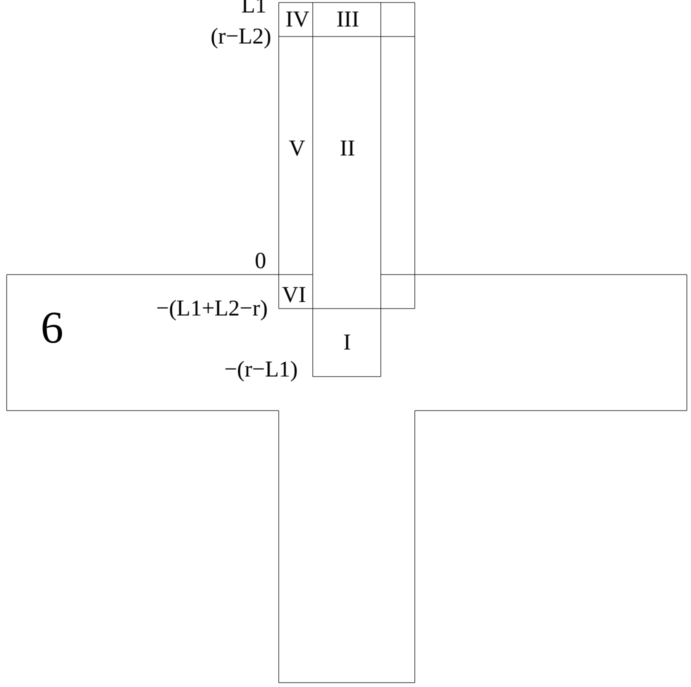

Figure 8: The cross regions for the interval $L1 + \frac{L2}{2} \leq r \leq L1 + L2$.

$$\mu_{VI}(r) = \frac{-100\,L^3 r + 90\,r^2 L^2 - 24\,L r^3 + 2\,r^4 + 49\,L^4}{9L^2}$$

$$\begin{aligned}\sigma^2_{VI}(r) \quad &= \quad -\frac{2}{81L^4}(11804\,L^6 r^2 + 6881\,L^4 r^4 + 464\,r^6 L^2 - 2360\,r^5 L^3 \\ &\quad -11766\,L^5 r^3 - 7042\,L^7 r - 48\,r^7 L + 2\,r^8 + 2273\,L^8)\end{aligned}$$

$$\Lambda_{VI}(r) = -\frac{L^2\,(4788\,L^4 r^2 + 666\,r^4 L^2 - 8\,r^6 - 3180\,L^3 r^3 - 4284\,L^5 r + 2145\,L^6)}{(-100\,L^3 r + 90\,r^2 L^2 - 24\,L r^3 + 2\,r^4 + 49\,L^4)^2}$$

2.2.7 Interval $L1 + L2 \leq r \leq 2L1 + L2$

$$\begin{aligned}A_I(x,y,r) &= 4L1L2 + L2^2 \quad x \in [0, L2], \quad y \in [-L2, 0] \\ A_{II}(x,y,r) &= 4L1L2 + L2^2 \quad x \in [0, L2], \quad y \in [0, r - L1 - L2] \\ A_{III}(x,y,r) &= 3L1L2 + L2^2 + (r - y - L2)L2 \quad x \in [0, L2], \quad y \in [r - L1 - L2, L1]\end{aligned}$$

$$\mu_{VII}(r) = \frac{31}{9}\,L^2 + \frac{20}{9}\,rL - \frac{2}{9}\,r^2$$

$$\sigma^2_{VII}(r) = -\frac{4\,(2\,L - r)\,(5\,L - r)^3}{81}$$

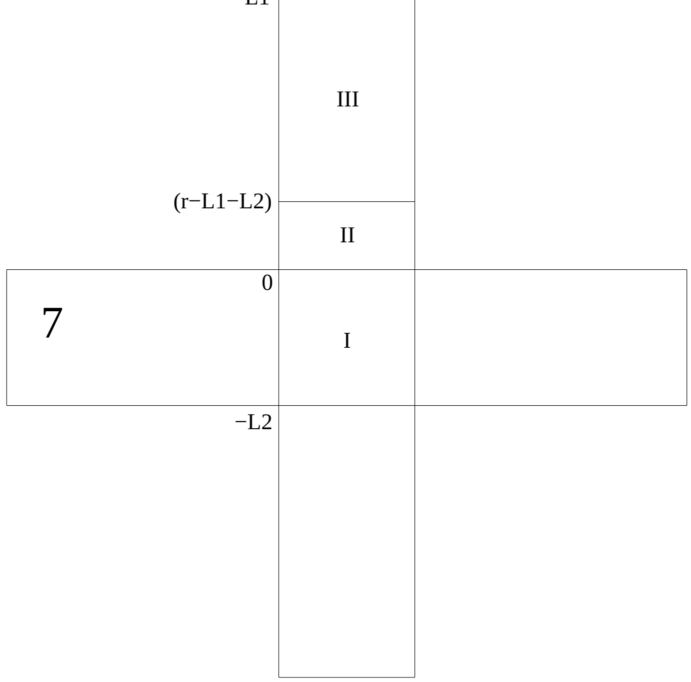

Figure 9: The cross regions for the interval $L1 + L2 \leq r \leq 2L1 + L2$.

$$\Lambda_{VII}(r) = -\frac{3L\left(13\,L^3 - 780\,rL^2 + 4\,r^3 + 48\,r^2L\right)}{\left(31\,L^2 + 20\,rL - 2\,r^2\right)^2}$$

Sumarizing the results for the cross, the whole expression for the lacunarity of a cross is given by the following expression

$$\Lambda(r) = \begin{cases} \Lambda_I(r) = \frac{4L2(-300Lr+59r2+324L2)}{(36L^2-20Lr+3r2)2} \\ \Lambda_{II}(r) = \frac{L^2\left(-12\,L^5r-192\,r^5L+452\,L^3r^3+L^6-36\,L^4r^2+12\,r^6+168\,r^4L^2\right)}{r^4\left(-20\,rL+36\,L^2+3\,r^2\right)^2} \\ \Lambda_{III}(r) = -3\,\frac{64\,r^4+320\,r^2L^2-348\,Lr^3-204\,L^3r+37\,L^4}{\left(12\,rL+L^2+6\,r^2\right)^2} \\ \Lambda_{IV}(r) = 3\,\frac{L\left(256\,r^2L+28\,r^3-228\,rL^2+71\,L^3\right)}{\left(12\,rL+L^2+6\,r^2\right)^2} \\ \Lambda_V(r) = \frac{L^2\left(1605\,L^6+5412\,L^4r^2+1662\,r^4L^2+40\,r^6-432\,r^5L-3320\,L^3r^3-4716\,L^5r\right)}{\left(-100\,L^3r+90\,r^2L^2-24\,Lr^3+2\,r^4+49\,L^4\right)^2} \\ \Lambda_{VI}(r) = -\frac{L^2\left(4788\,L^4r^2+666\,r^4L^2-8\,r^6-3180\,L^3r^3-4284\,L^5r+2145\,L^6\right)}{\left(-100\,L^3r+90\,r^2L^2-24\,Lr^3+2\,r^4+49\,L^4\right)^2} \\ \Lambda_{VII}(r) = -\frac{3L\left(13\,L^3-780\,rL^2+4\,r^3+48\,r^2L\right)}{\left(31\,L^2+20\,rL-2\,r^2\right)^2} \end{cases}$$

The corresponding intervals, I-VII, are shown in Tabel 2.2.7 and the resulting lacunarity curves for this piecewisely determined function is plotted in Figure 10.

Region	Interval
I	$r \in [0, \frac{L2}{2}]$
II	$r \in [\frac{L2}{2}, L2]$
III	$r \in [L2, \frac{L1+L2}{2}]$
IV	$r \in [\frac{L1+L2}{2}, L1]$
V	$r \in [L1, L1 + \frac{L2}{2}]$
VI	$r \in [L1 + \frac{L2}{2}, L1 + L2]$
VII	$r \in [L1 + L2, 2L1 + L2]$

Table 1: Regions for the analytical calculation of the self-referred lacunarity.

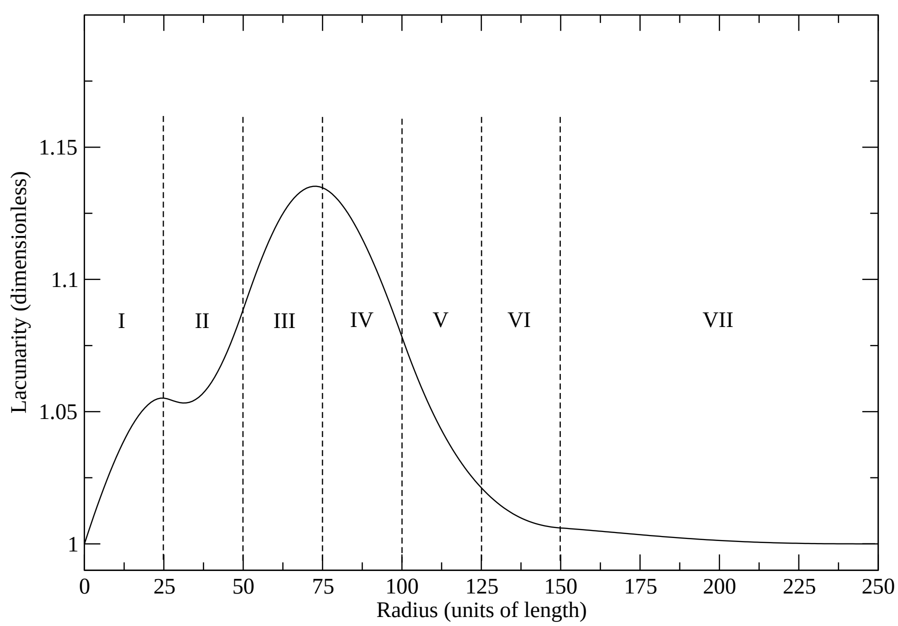

Figure 10: The analytically calculated self-referred lacunarity curve for a cross with dimensions $L1$ and $L2$ of 100 and 50 units of length respectively.